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Quantum Computational Semantics on Fock Space

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In the Fock space semantics, meanings of sentences are identified with density operators of the (unsymmetrized) Fock space $\mathcal{F}$ based on the Hilbert space $\mathbb{C}^2$. Generally, the meaning of a sentence is smeared over different sectors of $\mathcal{F}$. The standard quantum computational semantics is a limit case of the Fock space semantics, where the meaning of any sentence α only "lives" in one sector of $\mathcal{F}$, which is determined by the logical complexity of α. We prove that the *global* Fock space semantics and the standard quantum computational semantics characterize the same logic.

KEY WORDS: quantum computation; quantum gates; quantum logic; Fock space.
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1. INTRODUCTION

The theory of quantum logical gates in quantum computation has suggested new forms of quantum logic, called *quantum computational logics* (Dalla Chiara *et al.*, 2003). The basic semantic idea is the following: the meaning of a sentence α is identified with a *quantum information quantity*, represented by a density operator of a Hilbert space, whose dimension depends on the logical complexity of α. At the same time, the logical connectives are interpreted as logical operations defined in terms of quantum logical gates. In this framework, the sentences of the quantum computational language can be regarded as "economical and intuitive" descriptions of *quantum circuits* (Dalla Chiara *et al.*, 2003). The standard quantum computational semantics can be naturally generalized to a *Fock space semantics* (Gudder, 2004; Dalla Chiara *et al.*, 2004), where the meanings of all sentences

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"live" in a unique Fock space $\mathcal{F}$ and are generally smeared over different *sectors* of $\mathcal{F}$. From an intuitive point of view, the sectors of $\mathcal{F}$ can be regarded as different information-contexts which a sentence α can refer to; at the same time, the increasing number of particles described in the different sectors of $\mathcal{F}$ can be interpreted as *increasing information*.

2. THE BASIC DEFINITIONS

Consider the Hilbert space $\mathbb{C}^2$ (based on the set of all ordered pairs of complex numbers), and let $\mathcal{H}^{(k)}$ be the k-fold tensor product $\otimes^k\mathbb{C}^2$ (where $\mathcal{H}^{(0)} = \mathbb{C}$ and $\mathcal{H}^{(1)} = \mathbb{C}^2$). The canonical orthonormal basis $\mathbf{B}^{(k)}$ of $\mathcal{H}^{(k)}$ is defined as follows:

$$\mathbf{B}^{(0)} = \{1\},\ \mathbf{B}^{(1)} = \{|0\rangle, |1\rangle\},\ \mathbf{B}^{(k)} = \{|x_1, \ldots, x_k\rangle : x_1 \in \{0, 1\}, \ldots, x_k \in \{0, 1\}\},$$

where $k > 1$, $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$, while $|x_1, \ldots, x_k\rangle$ is an abbreviation for the tensor product $|x_1\rangle \otimes \ldots \otimes |x_k\rangle$. In order to stress that $|\psi\rangle$ is a vector of $\mathcal{H}^{(k)}$ we will also write $|\psi\rangle_{\mathcal{H}^{(k)}}$.

Definition 2.1. The Fock space on $\mathbb{C}^2$
The Fock space on $\mathbb{C}^2$ is the (infinite-dimensional) Hilbert space

$$\mathcal{F} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \ldots \oplus \mathcal{H}^{(k)} \oplus \ldots,$$

where $\oplus$ represents the direct sum. The following conditions are required:

(1) Any vector of $\mathcal{F}$ has the form:

$$|\psi\rangle = \left(\left|\psi^{(0)}\right\rangle, \left|\psi^{(1)}\right\rangle, \ldots, \left|\psi^{(k)}\right\rangle, \ldots\right),$$

where each $|\psi^{(k)}\rangle$ is a vector of $\mathcal{H}^{(k)}$ such that $\Sigma_k \| |\psi^{(k)}\rangle \|^2 < \infty$. The vector $|\psi^{(k)}\rangle$ is called the k-th *component* of $|\psi\rangle$, the space $\mathcal{H}^{(k)}$ is called the k-th *sector* of $\mathcal{F}$, while $\mathcal{H}^{(0)}$ represents the space for the *vacuum*.

(2) The sum and the scalar product of $\mathcal{F}$ are component-wise defined: the k-th component of $|\psi\rangle + |\varphi\rangle$ is $|\psi^{(k)}\rangle + |\varphi^{(k)}\rangle$ and the k-th component of $a|\psi\rangle$ is $a|\psi^{(k)}\rangle$.

(3) The inner product of $\mathcal{F}$ is defined as follows:

$$\langle\psi|\varphi\rangle = \sum_k \left\langle\psi^{(k)}\middle|\varphi^{(k)}\right\rangle.$$

Definition 2.2. The projection-functions.

- $\Pi^{(k)}$ associates to any vector of $\mathcal{F}$ its k-th component. In other words, for any $|\psi\rangle$ of $\mathcal{F} : \Pi^{(k)}(|\psi\rangle) = |\psi^{(k)}\rangle$.

- $\Pi^{(k,\mathcal{F})}$ associates to any vector of $\mathcal{H}^{(k)}$ its canonical extension to $\mathcal{F}$. In other words, for any $|\psi\rangle_{\mathcal{H}^{(k)}}$:

$$\Pi^{(k,\mathcal{F})}(|\psi\rangle_{\mathcal{H}^{(k)}}) = \left(\mathbf{0}^{(0)}, \mathbf{0}^{(1)}, \ldots, \mathbf{0}^{(k-1)}, |\psi\rangle_{\mathcal{H}^{(k)}}, \mathbf{0}^{(k+1)}, \ldots\right),$$

  where $\mathbf{0}^{(i)}$ is the null vector of the i-sector.

We also write $|\psi\rangle^{(k,\mathcal{F})}$ instead of $\Pi^{(k,\mathcal{F})}(|\psi\rangle_{\mathcal{H}^{(k)}})$.
The canonical orthonormal basis of $\mathcal{F}$ is the set

$$\mathbf{B}^{\mathcal{F}} = \{1\} \cup \left\{|x_1, \ldots, x_k\rangle^{(k,\mathcal{F})} : |x_1, \ldots, x_k\rangle \in \mathbf{B}^{(k)}\right\}_{k\geq 1}.$$

Definition 2.3. Quregisters.
A *quregister* is a unit vector of $\mathcal{H}^{(k)}$ (where $k \geq 1$).

Definition 2.4. Qumixes and Fock qumixes.

- A *qumix* is a density operator ρ of a Hilbert space $\mathcal{H}^{(k)}$.
- A *Fock qumix* is a density operator ρ of the Fock space $\mathcal{F}$.

Any operator $A^{(k)}$ of the sector $\mathcal{H}^{(k)}$ admits a canonical Fock-extension $A^{(k,\mathcal{F})}$, which is defined for any vector $|\psi\rangle$ of $\mathcal{F}$:

$$A^{(k,\mathcal{F})}(|\psi\rangle) = \left[A^{(k)}\left(\left|\psi^{(k)}\right|\right)\right]^{(k,\mathcal{F})}.$$

Hence, in particular, $\rho^{(k,\mathcal{F})}$ represents the canonical Fock-extension of the qumix ρ of the sector $\mathcal{H}^{(k)}$. At the same time, any Fock qumix ρ has, for any sector $\mathcal{H}^{(k)}$, a *k-restriction* and a *k-counterpart*:

- The *k-restriction* of ρ is the Fock operator $\rho^{\upharpoonright k} := I^{(k,\mathcal{F})}\rho I^{(k,\mathcal{F})}$, where $I^{(k)}$ is the identity of the k-sector.
- The *k-counterpart* of ρ is the operator of $\mathcal{H}^{(k)}$ that is defined as follows: $\rho^{(k)}(|\psi\rangle) := \Pi^{(k)}(\rho^{\upharpoonright k}(|\psi\rangle^{(k,\mathcal{F})}))$ (for any vector $|\psi\rangle$ of $\mathcal{H}^{(k)}$).

Both $\rho^{\upharpoonright k}$ and $\rho^{(k)}$ are positive trace-class operators such that $\mathtt{tr}(\rho^{\upharpoonright k}) \leq 1$ and $\mathtt{tr}(\rho^{(k)}) \leq 1$.

In order to introduce a convenient *quantum computational semantics*, we shall distinguish between *true* and *false* quregisters (which represent natural generalizations of the two classical truth-values 1 and 0).

Definition 2.5. True and false quregisters of $\mathcal{H}^{(k)}$ (with $k \geq 1$).

- A *true quregister* of $\mathcal{H}^{(k)}$ is a basis element having the form $|x_1, \ldots, x_{k-1}, 1\rangle$.
- A *false quregister* of $\mathcal{H}^{(k)}$ is a basis element having the form $|x_1, \ldots, x_{k-1}, 0\rangle$.

In this framework, one can define the projections that represent the *Truth* and the *Falsity* properties in any space $\mathcal{H}^{(k)}$ (with $k \geq 1$) and in $\mathcal{F}$.

Definition 2.6. The truth-projection and the falsity-projection of $\mathcal{H}^{(k)}$ (with $k \geq 1$).

- The *truth-projection* of $\mathcal{H}^{(k)}$ is the projection $P_1^{(k)}$ whose range is the closed subspace spanned by the set of all true quregisters of $\mathcal{H}^{(k)}$.
- The *falsity-projection* of $\mathcal{H}^{(k)}$ is the projection $P_0^{(k)}$ whose range is the closed subspace spanned by the set of all false quregisters of $\mathcal{H}^{(k)}$.

Definition 2.7. The truth-projection and the falsity-projection of $\mathcal{F}$.

$$P_1^{\mathcal{F}} = \oplus_k P_1^{(k)}; \quad P_0^{\mathcal{F}} = \oplus_k P_0^{(k)}$$

where $P_1^{(0)}$ is the null operator of $\mathbb{C}$ and $P_0^{(0)}$ is the identity operator of $\mathbb{C}$.

By recalling the Born rule, we can now define for any qumix ρ of $\mathcal{H}^{(k)}$, the probability that ρ is true in $\mathcal{H}^{(k)}$.

Definition 2.8. The probability of a qumix.

$$p(\rho) = \mathtt{tr}\big(P_1^{(k)}\rho\big).$$

Clearly, $p(\rho)$ represents the probability that a quantum system in state ρ satisfies the truth-property.

In the case of the Fock space $\mathcal{F}$, different notions of probability turn out to be significant.

Definition 2.9. Fock probabilities.

Let ρ be a Fock qumix.

- The *probability of being localized in the k-sector*
  $p_\rho(k) = \mathtt{tr}(I^{(k,\mathcal{F})}\rho)$.
- The *probability of being true in the k-sector (sectorial probability)*
  $p_k(\rho) = \mathtt{tr}(P_1^{(k)}\rho^{(k)})$.
- The *probability of being true (global probability)*
  $p(\rho) = \mathtt{tr}(P_1^{\mathcal{F}}\rho)$.

Lemma 2.1. *Let ρ be a Fock qumix.*

(i) for any $k \geq 0 : 0 \leq p_\rho(k) \leq 1$;
(ii) for any $k > 0 : 0 \leq p_k(\rho) \leq 1$, and $p_0(\rho) = 0$;
(iii) for any $k \geq 0$: $p_k(\rho) \leq p_\rho(k)$;
(iv) $p(\rho) = \sum_k p_k(\rho)$.

Consequently: $p_k(\rho) = 1 \Rightarrow \forall h \neq k : p_h(\rho) = 0$.

3. FOCK QUANTUM COMPUTATIONAL STRUCTURES

Quantum logical gates (briefly *gates*) are unitary operators that transform quregisters into quregisters. We will consider the following gates: the *negation*, the *square root of the negation*, the *square root of the identity* and the *Petri-Toffoli gate* (Petri, 1967).

Let us first define our gates on the sectors $\mathcal{H}^{(k)}$ such that $k \geq 1$.

Definition 3.1. The negation.
The *negation* is the linear operator $\mathtt{Not}^{(k)}$ that is defined on the basis $\mathbf{B}^{(k)}$ of $\mathcal{H}^{(k)}$ as follows:

$$\mathtt{Not}^{(k)}(|x_1, \ldots, x_k\rangle) = |x_1, \ldots, x_{k-1}, 1 - x_k\rangle.$$

Definition 3.2. The square root of the negation and the square root of the identity.

- The *square root of the negation* is the linear operator $\sqrt{\mathtt{Not}}^{(k)}$ that is defined on the basis $\mathbf{B}^{(k)}$ of $\mathcal{H}^{(k)}$ as follows:

$$\sqrt{\mathtt{Not}}^{(k)}(|x_1, \ldots, x_k\rangle) = |x_1, \ldots, x_{k-1}\rangle \otimes \frac{1}{2}((1+i)|x_k\rangle + (1-i)|1 - x_k\rangle),$$

  (where i is the imaginary unit).
- The *square root of the identity* is the linear operator $\sqrt{\mathtt{Id}}^{(k)}$ that is defined on the basis $\mathbf{B}^{(k)}$ of $\mathcal{H}^{(k)}$ as follows:

$$\sqrt{\mathtt{Id}}^{(k)}(|x_1, \ldots, x_k\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle).$$

  Clearly, $\sqrt{\mathtt{Id}}^{(1)}$ corresponds to the Hadamard operator.

The negation is an example of a *semiclassical gate* that transforms classical registers into classical registers. The square root of the negation and the square root of the identity are important examples of *genuine quantum gates* that transform classical registers into proper quregisters (Nielsen *et al.*, 2000).

For any quregister $|\psi\rangle$ of $\mathcal{H}^{(k)}$ we have:

$$\sqrt{\mathtt{Not}}^{(k)}(\sqrt{\mathtt{Not}}^{(k)}(|\psi\rangle)) = \mathtt{Not}(|\psi\rangle); \sqrt{\mathtt{Id}}^{(k)}(\sqrt{\mathtt{Id}}^{(k)}(|\psi\rangle)) = |\psi\rangle.$$

¿From an intuitive point of view, $\sqrt{\mathtt{Not}}^{(k)}$ and $\sqrt{\mathtt{Id}}^{(k)}$ can be regarded as a "tentative partial negation" and a "tentative partial assertion" (respectively), that transform precise pieces of information into maximally uncertain ones (physical models of $\sqrt{\mathtt{Not}}^{(k)}$ have been described in Dalla Chiara *et al.* (2003)). Interestingly enough, both gates do not admit either a Boolean or a continuous fuzzy counterpart.

Lemma 3.1.

(1) There is no continuous function $f : [0, 1] \to [0, 1]$ *such that for any* $x \in [0, 1] : f(f(x)) = 1 - x$.
(2) There is no continuous function $f : [0, 1] \to [0, 1]$ *such that for any* $x \in [0, 1] : f(f(x)) = x,\ f(x) \neq x,\ f(x) \neq 1 - x$.

Proof: (1) has been proved in Dalla Chiara *et al*. (2003). Let us sketch the proof of (2) (by contradiction). By hypothesis, $f(0) \neq 0$ and $f(f(0)) = 0$. By continuity, there exist $x_1, x_2 \in [0, 1]$ such that $x_1 < f(0) < x_2$ and $f(x_1) = f(x_2)$. But $f(f(x_1)) = x_1 \neq x_2 = f(f(x_2))$, contradiction. □

Definition 3.3. The Petri-Toffoli gate.
For any $m, n \geq 1$ the *Petri-Toffoli gate* is the linear operator $\mathtt{T}^{(m,n,1)}$ that is defined on the basis $\mathbf{B}^{(m+n+1)}$ of $\mathcal{H}^{(m+n+1)}$ as follows:

$$\mathtt{T}^{(m,n,1)}(|x_1, \ldots, x_m, y_1, \ldots, y_n, z\rangle) = |x_1, \ldots, x_m, y_1, \ldots, y_n, (x_m y_n) \boxplus z\rangle,$$

where $\boxplus$ is the sum modulo 2.

One can prove that the negation, the square root of the negation, the square root of the identity and, the Petri-Toffoli gate are unitary operators. All these gates can be naturally generalized to the Fock space $\mathcal{F}$.

Definition 3.4. The Fock negations and the Fock square root of the identity.

- $\mathtt{Not}^{\mathcal{F}} := \oplus_k \mathtt{Not}^{(k)}$;
- $\sqrt{\mathtt{Not}}^{\mathcal{F}} := \oplus_k \sqrt{\mathtt{Not}}^{(k)}$;
- $\sqrt{\mathtt{Id}}^{\mathcal{F}} := \oplus_k \sqrt{\mathtt{Id}}^{(k)}$,
  where $\mathtt{Not}^{(0)}$, $\sqrt{\mathtt{Not}}^{(0)}$ and $\sqrt{\mathtt{Id}}^{(0)}$ are the identity operator of $\mathbb{C}$.

Definition 3.5. The Fock Petri-Toffoli gate.

$$\mathtt{T}^{\mathcal{F}(m,n,1)} := I^{(0)} \oplus \ldots \oplus I^{(m+n)} \oplus T^{(m,n,1)} \oplus I^{(m+n+2)} \oplus \ldots$$

where $T^{(0,n,1)}$ and $T^{(m,0,1)}$ are the identity operator of $\mathcal{H}^{(n+1)}$ and $\mathcal{H}^{(m+1)}$, respectively.

On can prove that $\mathtt{Not}^{\mathcal{F}}$, $\sqrt{\mathtt{Not}}^{\mathcal{F}}$, and $\mathtt{T}^{\mathcal{F}(m,n,1)}$ are unitary operators.

The gates considered so far can be naturally generalized to qumixes and to Fock qumixes. Since one is dealing with different mathematical objects, we will use different symbols.

Definition 3.6. Qumix-logical operations.
Let $k \geq 1$.

- The negation
  For any qumix ρ of $\mathcal{H}^{(k)}$, $\mathtt{NOT}^{(k)}(\rho) := \mathtt{Not}^{(k)}\rho\,\mathtt{Not}^{(k)}$.
- The square root of the negation
  For any qumix ρ of $\mathcal{H}^{(k)}$, $\sqrt{\mathtt{NOT}}^{(k)}(\rho) := \sqrt{\mathtt{Not}}^{(k)}\rho\,\sqrt{\mathtt{Not}}^{(k)*}$ (where $\sqrt{\mathtt{Not}}^{(k)*}$ is the adjoint of $\sqrt{\mathtt{Not}}^{(k)}$).
- The square root of the identity
  For any qumix ρ of $\mathcal{H}^{(k)}$, $\sqrt{\mathtt{ID}}^{(k)}(\rho) := \sqrt{\mathtt{Id}}^{(k)}\rho\,\sqrt{\mathtt{Id}}^{(k)}$.
- The Petri-Toffoli operation
  Let $m, n \geq 1$. For any qumixes ρ of $\mathcal{H}^{(m)}$, σ of $\mathcal{H}^{(n)}$ and τ of $\mathcal{H}^{(1)}$,
  $$\mathbb{T}^{(m,n,1)}(\rho, \sigma, \tau) := \mathtt{T}^{(m,n,1)}(\rho \otimes \sigma \otimes \tau)\mathtt{T}^{(m,n,1)}.$$

In a similar way, we can define the corresponding Fock logical operations.

Definition 3.7. Fock logical operations.

- The negation
  For any Fock qumix ρ, $\mathtt{NOT}^{\mathcal{F}}(\rho) := \mathtt{Not}^{\mathcal{F}}\rho\,\mathtt{Not}^{\mathcal{F}}$.
- The square root of the negation
  For any Fock qumix ρ, $\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\rho) := \sqrt{\mathtt{Not}}^{\mathcal{F}}\rho\,\sqrt{\mathtt{Not}}^{\mathcal{F}*}$.
- The square root of the identity
  For any Fock qumix ρ, $\sqrt{\mathtt{ID}}^{\mathcal{F}}(\rho) := \sqrt{\mathtt{Id}}^{\mathcal{F}}\rho\,\sqrt{\mathtt{Id}}^{\mathcal{F}}$.
- The Petri-Toffoli operation
  For any Fock qumixes ρ, σ and τ,
  $$\mathbb{T}^{\mathcal{F}}(\rho, \sigma, \tau) := \sum_{m\geq 0, n\geq 0} \mathtt{T}^{\mathcal{F}(m,n,1)}\big(\rho^{(m,\mathcal{F})} \otimes \sigma^{(n,\mathcal{F})} \otimes \tau^{(1,\mathcal{F})}\big)\mathtt{T}^{\mathcal{F}(m,n,1)}.$$

One can prove that all the logical operations we have defined transform qumixes into qumixes. The following theorem describes some basic properties of the Fock operations.

Theorem 3.2. *For any Fock qumix ρ, σ such that $p_\rho(0) = 0$ and $p_\sigma(0) = 0$:*

- *(i)* $p(\mathtt{NOT}^{\mathcal{F}}(\rho)) = 1 - p(\rho)$;
- *(ii)* $p(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\mathbb{T}^{\mathcal{F}}(\rho, \sigma, P_0^{(1,\mathcal{F})}))) = \frac{1}{2}$;
- *(iii)* $p(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\mathbb{T}^{\mathcal{F}}(\rho, \sigma, P_0^{(1,\mathcal{F})}))) = \frac{1}{2}$;
- *(iv)* $p(\mathbb{T}^{\mathcal{F}}(\rho, \sigma, P_0^{(1,\mathcal{F})})) = p(\rho)p(\sigma)$;

- *(v)* $p(\mathtt{NOT}^{\mathcal{F}}(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\rho))) = p(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\mathtt{NOT}^{\mathcal{F}}(\rho)))$;
- *(vi)* $p(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\rho))) = p(\mathtt{NOT}^{\mathcal{F}}(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\rho)))$;
- *(vii)* $p(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\mathtt{NOT}^{\mathcal{F}}(\rho))) = p(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\rho))$;
- *(viii)* $p(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\rho))) = p(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\rho))$.

Proof: Slight modification of the proof given for finite-dimensional Hilbert spaces in Gudder (2003, 2004). □

The set $\mathcal{D}(\mathcal{F})$ of all Fock qumixes can be naturally preordered by two relations that we call the *global* and the *sectorial preorders*, respectively.

Definition 3.8. Global preorder.
$\rho \preccurlyeq^{\mathrm{Glob}} \sigma$ iff

$$p(\rho) \leq p(\sigma) \quad \text{and} \quad p\big(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\sigma)\big) \leq p\big(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\rho)\big) \quad \text{and} \quad p\big(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\rho)\big) \leq p(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\sigma)).$$

Definition 3.9. Sectorial preorder.
$\rho \preccurlyeq^{\mathrm{Sec}} \sigma$ iff

$$\forall k \geq 0 \quad [\ p\big(\rho^{(k)}\big) \leq p\big(\sigma^{(k)}\big) \quad \text{and} \quad p\big(\sqrt{\mathtt{NOT}}^{(k)}\big(\sigma^{(k)}\big)\big) \leq p\big(\sqrt{\mathtt{NOT}}^{(k)}\big(\rho^{(k)}\big)\big) \quad \text{and} \quad p\big(\sqrt{\mathtt{ID}}^{(k)}\big(\rho^{(k)}\big)\big) \leq p\big(\sqrt{\mathtt{ID}}^{(k)}\big(\sigma^{(k)}\big)\big)].$$

On this basis we can define two *Fock quantum computational structures*:

Definition 3.10.

(1) *The global Fock quantum computational structure*

$$\langle \mathcal{D}(\mathcal{F}),\ \preccurlyeq^{\mathrm{Glob}},\ \mathtt{NOT}^{\mathcal{F}},\ \sqrt{\mathtt{NOT}}^{\mathcal{F}},\ \sqrt{\mathtt{ID}}^{\mathcal{F}},\ \mathbb{T}^{\mathcal{F}} \rangle.$$

(2) *The sectorial Fock quantum computational structure*

$$\langle \mathcal{D}(\mathcal{F}),\ \preccurlyeq^{\mathrm{Sec}},\ \mathtt{NOT}^{\mathcal{F}},\ \sqrt{\mathtt{NOT}}^{\mathcal{F}},\ \sqrt{\mathtt{ID}}^{\mathcal{F}},\ \mathbb{T}^{\mathcal{F}} \rangle.$$

The two preorders $\preccurlyeq^{\mathrm{Glob}}$ and $\preccurlyeq^{\mathrm{Sec}}$ naturally determine two equivalence relations (defined on $\mathcal{D}(\mathcal{F})$):

- $\rho \equiv^{\mathrm{Glob}} \sigma$ iff $\rho \preccurlyeq^{\mathrm{Glob}} \sigma$ and $\sigma \preccurlyeq^{\mathrm{Glob}} \rho$.
- $\rho \equiv^{\mathrm{Sec}} \sigma$ iff $\rho \preccurlyeq^{\mathrm{Sec}} \sigma$ and $\sigma \preccurlyeq^{\mathrm{Sec}} \rho$.

Let $\equiv$ represent either $\equiv^{\text{Glob}}$ or $\equiv^{\text{Sec}}$. One can easily show that $\equiv$ is a congruence-relation with respect to the Fock logical operations $\mathtt{NOT}^{\mathcal{F}}$, $\sqrt{\mathtt{NOT}}^{\mathcal{F}}$, $\sqrt{\mathtt{ID}}^{\mathcal{F}}$,$\mathbb{T}^{\mathcal{F}}$ (in other words, $\rho \equiv \sigma$ implies $\mathtt{NOT}^{\mathcal{F}}(\rho) \equiv \mathtt{NOT}^{\mathcal{F}}(\sigma)$; and in a similar way in the case of $\sqrt{\mathtt{NOT}}^{\mathcal{F}}$, $\sqrt{\mathtt{ID}}^{\mathcal{F}}$,$\mathbb{T}^{\mathcal{F}}$). As a consequence, we obtain two quotient structures that we call *contracted Fock quantum computational structures*:

- $\langle [\mathcal{D}(\mathcal{F})]_{\equiv^{\text{Glob}}}, \mathtt{NOT}^{\mathcal{F}}, \sqrt{\mathtt{NOT}}^{\mathcal{F}}, \sqrt{\mathtt{ID}}^{\mathcal{F}}, \mathbb{T}^{\mathcal{F}} \rangle$.
- $\langle [\mathcal{D}(\mathcal{F})]_{\equiv^{\text{Sec}}}, \mathtt{NOT}^{\mathcal{F}}, \sqrt{\mathtt{NOT}}^{\mathcal{F}}, \sqrt{\mathtt{ID}}^{\mathcal{F}}, \mathbb{T}^{\mathcal{F}} \rangle$,

where the Fock operations are defined on the equivalence classes of both structures in the expected way (for instance, $\mathtt{NOT}^{\mathcal{F}}([\sigma]_{\equiv^{\text{Glob}}}) := [\mathtt{NOT}^{\mathcal{F}}(\sigma)]_{\equiv^{\text{Glob}}}$), and in a similar way in the other cases).

4. FOCK QUANTUM COMPUTATIONAL LOGICS

In the Fock space semantics any sentence α of the quantum computational language $\mathcal{L}^{\text{QC}}$ is interpreted as a qumix of the Fock space $\mathcal{F}$, while the logical connectives are interpreted as the corresponding Fock logical operations. The language $\mathcal{L}^{\text{QC}}$ contains atomic sentences, a privileged atomic sentence $\mathbf{f}$ (whose intended interpretation is the truth-value *Falsity*) and the following primitive connectives: the *negation* ($\neg$), the *square root of the negation* ($\sqrt{\neg}$), the *square root of the identity* ($\sqrt{id}$), a ternary *conjunction* $\wedge$ (which corresponds to the Petri-Toffoli gate). For any sentences α and β, the expression $\wedge(\alpha, \beta, \mathbf{f})$ is a sentence of $\mathcal{L}^{\text{QC}}$. In this framework, the usual conjunction $\alpha \wedge \beta$ is dealt with as the metalinguistic abbreviation for the ternary conjunction $\wedge(\alpha, \beta, \mathbf{f})$. For any sentence α, the number of atomic sentences occurring in α is called the *atomic complexity* of α (abbreviated as $At(\alpha)$).

Definition 4.1. Fock Model.

A *Fock model* is a function $\mathtt{Qum}^{\mathcal{F}}$ that associates to any sentence α a Fock qumix such that the following conditions hold:

- $p_{\mathtt{Qum}^{\mathcal{F}}(\alpha)}(0) = 0$ for any atomic sentence α;
- $\mathtt{Qum}^{\mathcal{F}}(\mathbf{f}) = P_0^{(1,\mathcal{F})}$;
- $\mathtt{Qum}^{\mathcal{F}}(\neg\beta) = \mathtt{NOT}^{\mathcal{F}}(\mathtt{Qum}^{\mathcal{F}}(\beta))$;
- $\mathtt{Qum}^{\mathcal{F}}(\sqrt{\neg}\,\beta) = \sqrt{\mathtt{NOT}}^{\mathcal{F}}(\mathtt{Qum}^{\mathcal{F}}(\beta))$;
- $\mathtt{Qum}^{\mathcal{F}}(\sqrt{id}\,\beta) = \sqrt{\mathtt{ID}}^{\mathcal{F}}(\mathtt{Qum}^{\mathcal{F}}(\beta))$;
- $\mathtt{Qum}^{\mathcal{F}}(\bigwedge(\alpha, \beta, \mathbf{f})) = \mathbb{T}^{\mathcal{F}}(\mathtt{Qum}^{\mathcal{F}}(\alpha), \mathtt{Qum}^{\mathcal{F}}(\beta), \mathtt{Qum}^{\mathcal{F}}(\mathbf{f}))$.

Definition 4.2. Sharply sectorial model.

A Fock model $\mathtt{Qum}^{\mathcal{F}}$ is *sharply sectorial* iff for any sentence α:

$$k \neq At(\alpha) \Rightarrow p_{\mathtt{Qum}^{\mathcal{F}}(\alpha)}(k) = 0.$$

The standard quantum computational semantics can be described as a limit-case of the Fock space semantics, where any model is sharply sectorial. In other words, the meaning of a sentence α only "lives" in the sector that corresponds to the atomic complexity of α.

As happens in the standard quantum computational semantics, also the Fock semantics gives rise to natural definitions of different *consequence relations* that permit us to semantically characterize different forms of *quantum computational logic*.[6]

Definition 4.3. Fock consequences in a model $\mathtt{Qum}^{\mathcal{F}}$.

- *Global consequence*
  $\alpha \models^{\mathrm{Glob}}_{\mathtt{Qum}^{\mathcal{F}}} \beta$ iff $\mathtt{Qum}^{\mathcal{F}}(\alpha) \preccurlyeq^{\mathrm{Glob}} \mathtt{Qum}^{\mathcal{F}}(\beta)$.
- *Sectorial consequence*
  $\alpha \models^{\mathrm{Sec}}_{\mathtt{Qum}^{\mathcal{F}}} \beta$ iff $\mathtt{Qum}^{\mathcal{F}}(\alpha)) \preccurlyeq^{\mathrm{Sec}} \mathtt{Qum}^{\mathcal{F}}(\beta))$.

Lemma 4.1. *If* $\alpha \models^{\mathrm{Sec}}_{\mathtt{Qum}^{\mathcal{F}}} \beta$, *then* $\alpha \models^{\mathrm{Glob}}_{\mathtt{Qum}^{\mathcal{F}}} \beta$, *but not conversely.*

Proof: By definition of global and sectorial consequence, and because $p(\rho) = \Sigma_k p_k(\rho)$. □

Lemma 4.2. *For any sharply sectorial model the consequence relations* $\models^{\mathrm{Glob}}_{\mathtt{Qum}^{\mathcal{F}}}$ *and* $\models^{\mathrm{Sec}}_{\mathtt{Qum}^{\mathcal{F}}}$ *collapse into one and the same relation.*

Proof: By definition of sharply sectorial model, of global and sectorial consequence, and because $p(\rho) = \Sigma_k p_k(\rho)$. □

We call the *Global Fock quantum computational logic* ($\mathbf{QCL}^{\mathcal{F}_{\mathrm{Glob}}}$) and the *Sectorial Fock quantum computational logic* ($\mathbf{QCL}^{\mathcal{F}_{\mathrm{Sec}}}$) the logics that are characterized by $\models^{\mathrm{Glob}}_{\mathtt{Qum}^{\mathcal{F}}}$ and $\models^{\mathrm{Sec}}_{\mathtt{Qum}^{\mathcal{F}}}$, respectively. Accordingly, we have:

- $\alpha \models_{\mathbf{QCL}^{\mathcal{F}_{\mathrm{Glob}}}} \beta$ iff for any $\mathtt{Qum}^{\mathcal{F}} : \alpha \models^{\mathrm{Glob}}_{\mathtt{Qum}^{\mathcal{F}}} \beta$;
- $\alpha \models_{\mathbf{QCL}^{\mathcal{F}_{\mathrm{Sec}}}} \beta$ iff for any $\mathtt{Qum}^{\mathcal{F}} : \alpha \models^{\mathrm{Sec}}_{\mathtt{Qum}^{\mathcal{F}}} \beta$.

Theorem 4.3.
$\alpha \models_{\mathbf{QCL}^{\mathcal{F}_{\mathrm{Glob}}}} \beta$ *iff for any sharply sectorial model* $\mathtt{Qum}^{\mathcal{F}}$, $\alpha \models_{\mathtt{Qum}^{\mathcal{F}}} \beta$.

[6] *Characterizing semantically* a logic **L** means defining a semantic relation that represents the *logical consequence* relation for **L**.

Proof: (Sketch)

The left to right implication is trivial. Suppose that $\alpha \nvDash_{QCL^{\mathcal{F}_{\mathrm{Glob}}}} \beta$. Hence, there exists a model $\mathtt{Qum}^{\mathcal{F}}$ s.t. $\mathtt{Qum}^{\mathcal{F}}(\alpha) \not\preccurlyeq^{\mathrm{Glob}} \mathtt{Qum}^{\mathcal{F}}(\beta)$. We prove that there exists a sharply sectorial model $\mathtt{Qum}_S^{\mathcal{F}}$ s.t. $\mathtt{Qum}_S^{\mathcal{F}}(\alpha) \not\preccurlyeq^{\mathrm{Glob}} \mathtt{Qum}_S^{\mathcal{F}}(\beta)$.

For any atomic sentence **q** consider the following real numbers:

$$r_1 = p(\mathtt{Qum}^{\mathcal{F}}(\mathbf{q})),\ r_2 = p\big(\sqrt{\mathtt{NOT}}^{\mathcal{F}}(\mathtt{Qum}^{\mathcal{F}}(\mathbf{q}))\big),$$
$$r_3 = p\big(\sqrt{\mathtt{ID}}^{\mathcal{F}}(\mathtt{Qum}^{\mathcal{F}}(\mathbf{q}))\big).$$

One can show that the triplet (r_1, r_2, r_3) uniquely determines a density operator $\rho^{(r_1,r_2,r_3)}$ of $\mathbb{C}^2$ s.t.:

$$p\big(\rho^{(r_1,r_2,r_3)}\big) = r_1,\ p\big(\sqrt{\mathtt{NOT}}^{(1)}\big(\rho^{(r_1,r_2,r_3)}\big)\big) = r_2,$$
$$\times\, p\big(\sqrt{\mathtt{ID}}^{(1)}\big(\rho^{(r_1,r_2,r_3)}\big)\big) = r_3.$$

We define $\mathtt{Qum}_S^{\mathcal{F}}$ as follows: for any atomic sentence $\mathbf{q}\mathtt{Qum}_S^{\mathcal{F}}(\mathbf{q}) = \rho^{(r_1,r_2,r_3)^{(1,\mathcal{F})}}$, where $r_1 = p(\mathtt{Qum}^{\mathcal{F}}(\mathbf{q}))$, $r_2 = p(\sqrt{\mathtt{NOT}}(\mathtt{Qum}^{\mathcal{F}}(\mathbf{q})))$, $r_3 = p(\sqrt{\mathtt{ID}}(\mathtt{Qum}^{\mathcal{F}}(\mathbf{q})))$. Apparently, $\mathtt{Qum}_S^{\mathcal{F}}$ is a sharply sectorial model s.t. for any atomic sentence **q**, $p(\mathtt{Qum}_S^{\mathcal{F}}(\mathbf{q})) = p(\mathtt{Qum}^{\mathcal{F}}(\mathbf{q}))$, $p(\mathtt{Qum}_S^{\mathcal{F}}(\sqrt{\neg}\mathbf{q})) = p(\mathtt{Qum}^{\mathcal{F}}(\sqrt{\neg}\mathbf{q}))$, $p(\mathtt{Qum}_S^{\mathcal{F}}(\sqrt{id}\,\mathbf{q})$ $= \mathtt{Qum}^{\mathcal{F}}(\sqrt{id}\,\mathbf{q}))$. □

Lemma 4.4. *For any sentence α, $p(\mathtt{Qum}_S^{\mathcal{F}}(\alpha)) = p(\mathtt{Qum}^{\mathcal{F}}(\alpha))$.*

Proof: By induction on the length of α, by definition of $\mathtt{Qum}_S^{\mathcal{F}}$ and by Theorem 3.2. □

Theorem 4.3 is an immediate consequence of Lemma 4.4.

Hence, the global Fock semantics and the standard quantum computational semantics characterize the same logic. One is dealing with a nonstandard form of quantum logic. Conjunctions and disjunctions do not correspond to lattice operations, because they are not generally idempotent ($\alpha \nvDash_{\mathbf{QCL}^{\mathcal{F}_{\mathrm{Glob}}}} \alpha \wedge \alpha$). Unlike Birkhoff and von Neumann's quantum logic, the weak distributivity principle breaks down: $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \nvDash_{\mathbf{QCL}^{\mathcal{F}_{\mathrm{Glob}}}} \alpha \wedge (\beta \vee \gamma)$. At the same time, the strong distributivity (which is violated in orthodox quantum logic) is valid here: $\alpha \wedge (\beta \vee \gamma) \models_{\mathbf{QCL}^{\mathcal{F}_{\mathrm{Glob}}}} (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$. Both the excluded middle and the noncontradiction principles are violated: $\neg\mathbf{f} \nvDash_{\mathbf{QCL}^{\mathcal{F}_{\mathrm{Glob}}}} \alpha \vee \neg\alpha$ and $\neg\mathbf{f} \nvDash_{\mathbf{QCL}^{\mathcal{F}_{\mathrm{Glob}}}} \neg(\alpha \wedge \neg\alpha)$. As a consequence, one can say that this form of quantum computational logic represents an example of *fuzzy logic*.

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